Association of Teachers of Mathematics

Thinkers

a collection of activities to provoke mathematical thinking

THINKERS

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Contents

This booklet sets out to provide ideas for classroom activities that stimulate mathematical thinking. All of the prompts ask learners to construct or deal with some mathematical object(s) which exemplify a concept or technique. We see exemplification and generalisation as characteristics of mathematical thought that are at the heart of 'doing mathematics'. We also know that everyone who gets to school has already displayed the power to generalise and to exemplify. The question is how to prompt learners to use those powers in mathematics lessons. In order to provide opportunities for learners to engage in mathematical thought we lay out some starting points for classroom discussion. Teachers can use the variety of 'generalisation-generating' activities to encourage learners to generalise. Indeed, in some of these, learners cannot avoid generalising, although they may need support in articulating their generalisations verbally or expressing them in writing, diagrams or symbols.

In short, we suggest that asking learners to generate examples for themselves can provide the foundations for recognition, articulation and appreciation of a generality. Teacher-generated examples always illustrate the teacher's understanding, and sometimes seem, to the learner, as if they are chosen at random. On the other hand encouraging learners to examine universalities and differences in an 'example space', created in the classroom, can cause them to notice the variation within that space and to express what is invariant. Classroom talk that focuses on 'same and different' (similarities and dissimilarities, commonalities and peculiarities – however it is described) concentrates on both concept development and the processes of abstraction. This talk, at its most mundane, can lead to algebraic symbolisation of 'rules'; at its most profound it can provide contact with the essence of mathematics.

HOW DO WE GENERALISE USEFULLY IN MATHEMATICS?

If young learners have only ever seen triangles which are equilateral, or which come in different colours, or are always plastic, the generalisation of triangles is unlikely to include different shapes. Nor is it likely to take into account that colour and thickness are not mathematical properties in this case. Learners need tasks which help them discriminate mathematically, and which help them broaden their example spaces.

It is by seeing what is different mathematically, but still has the same label, that we begin to notice important similarities. For example, if we only ever see teachers work out what x is when $2x + 1 = 7$ we may never grasp the general processes which are being taught (particularly if we already know that x has to be 3 from our own sense-making methods!) However, if there are lots of different-looking examples around which are all restatements of $x = 3$, there is something interesting to be sorted out. Furthermore, if there are lots of examples around which look a bit like $2x + 1 = 7$ but have different underlying values for x, there is something else of interest to be sorted out. However learners make sense of it, the chances of them making mathematical distinctions are very high, because it is mathematical variation which is being offered.

To design the tasks below we have used the fact that any mathematical topic, structure, statement, or expression contains a choice of dimensions of variation, and each dimension has an associated range of permissible change. Typically, learners play safe with ranges of change, not going beyond the familiar unless encouraged and supported in doing so; some of the prompts we offer work on this aspect of mathematical creativity. Learners may also not recognise the same dimensions of variation as the teacher: for example the teacher may think of varying the angles of a triangle but the learner may think of varying the line-width. Some of the prompts below work on expanding the learner's repertoire of variation.

Consider creating a diagram for learners to reflect objects in a mirror line; how many dimensions of possible variation are there? Firstly, should it be a drawing of a familiar object, an abstract shape, a geometrically-defined shape, or what? Should it consist of straight lines only, or some curved ones? Should the mirror line be horizontal, vertical, or neither? Should the mirror line be on the edge of the shape, through the shape, away from the shape, parallel to one edge, or none of these? Should the whole diagram be on a coordinate grid, with vertices only on integer coordinates, or not? Any answers to these questions could be valid depending on the purpose. Which would you choose if you wanted learners to sort out the relationship between reflective transformation and symmetrical shapes? As a teacher, you choose the dimensions of variation you want to present, but learners may be thinking about quite different dimensions of variation. Prompts like the ones in this booklet both reveal what learners are aware of, and prompt them to extend their horizons. By choice of prompt you can deliberately and successfully direct learner attention where you think is most appropriate mathematically.

If you deliberately ask learners to create their own examples by varying the dimension you want them to consider, or by pushing them beyond their safe ranges of change, you can be more sure they will experience the structures you want them to learn about. If you want them to develop general strategies for finding reflections wherever the mirror line is, then getting them to explore reflections of the same shape, but with different mirror lines, seems a sensible way to start. They will almost be compelled to generalise in order to make sense of the variety of different outcomes they produce. Our view is that there is no need to teach 'generalisation'; it is what we all do all the time anyway.

THE ACTIVITIES

The activities we suggest run across attainment targets and topics and through key stages. We have tried to give suggestions for all strands of mathematics. We hope that readers will appreciate that this scope will mean that not all examples are immediately applicable in their own teaching environment but that it does indicate the applicability of the approach at all levels. Our examples are intended to illustrate sixteen different classes of questions, tasks or prompts all of which invite learners to engage directly with the subject matter of mathematics. You can use them as starting points which will stimulate your own ideas to suit the class and the topic you are teaching.

The questions, tasks or prompts under each of the sixteen headings have been arranged in an approximation to the order in which they might be met in schools from KS2 to AS/A2 level. Teachers of older learners might usefully use some of the 'easier' activities to help develop learners' abilities to generalise first in topics with which they are familiar. Teachers of younger learners might use some activities related to areas of mathematics not yet taught in order to challenge and stimulate. They are numbered simply for the purpose of indexing – they do not have to be done in any particular order.

After illustrating the sixteen different kinds of question, task or prompt, in "Using 'Thinkers' in Shape and Space", we give an example of a task sequence which could contribute to a learner's progression towards formal geometric proof. This sequence could be used across key stages, because what is being developed is the learner's ability to create and question in mathematics by generalising from examples.

We refer to the sixteen types by these names:

1. Give An Example Of … (another and another)
2. Pointing Toward Generality (particular, peculiar, general)
3. Hard and Easy
4. Additional Conditions
5. Comparing/Contrasting three
6. Confounding Expectations
7. Impossible Constructions
8. On the Spot Generalisation
9. Open and Closed Questions / Exhaustive lists
10. Always, Sometimes, Never true
11. Odd One Out
12. Sorting
13. Ordering
14. Equivalent Statements
15. With and Across the Grain
16. Burying the Bone

Naming types of questions can be useful because the label may be brought to mind in the midst of some situation, affording access to its specific type of question. At first, our label may be alien, and you may wish to replace it with one of your own; once a label has become connected with a rich collection of instances and experience of using those questions, the label serves to trigger those questions when you need them. Labels also enable teachers and learners to reflect on, and communicate with each other about the use of those questions, about differences and similarities in the thinking required.

1 Give An Example Of … (another and another)

The instruction 'Give an example of …' starts with a generalisation and asks learners to provide particular examples. Learners are then asked to say what is common about all the responses. More examples can be generated, and more thinking, by asking first for one example, (pause for construction) then another (pause for construction) then another.

Give, find, construct an example of:

1. a pair of numbers that differ by 2; and another; and another.

The task format of '… and another and another …' provokes a variety of reactions. Some people write whatever comes to mind first, some have a systematic approach to generating the second and third example, some make the examples progressively more complex, some seek to generalise.

As teachers we can help develop learners' ability to generalise by emphasising not only what is common to all the learner-generated examples but also asking how they were generated.

We could ask after each of these "How do you create your examples?" So for instance "How do you create a pair of numbers that differ by 2?"

Some people might start with a number and add 2, so x and $x+2$ differ by 2.
Or we could start with a number and subtract 2, so x and $x - 2$ differ by 2.
Or we could go one either side of a number, so $x - 1$ and $x + 1$ differ by 2.
Or …

It is also worth considering that '… a difference of 2' may not provoke generalisation (most people just 'know' that two numbers differ by 2 without thinking about it) – try 'a pair of numbers that differ by 2.68'. You may find yourself having to think more about how to generate a pair. So this can lead to a generalisation more easily than the 'easier' question.

We offer multiple variations of this particular form of question in order to illustrate how the other 15 types might be varied in a similar way.

Teachers can create new starting points by applying the 'what if not …?' process. It is worth making this process explicit to the learners in order to open the dialogue on dimensions of variation. 'What can be varied in the original question?' used as a frequent question to stimulate the problem posing aspect of problem solving will develop the notion of variation. So:

What if not 'differ'?

Give, find, construct an example of:

2. a pair of numbers whose **sum** is 2; and another, and another.
3. a pair of numbers whose **product** is 2; and another, and another.
4. a pair of numbers whose **quotient** is 2; and another, and another.

What if not 2?

Give, find, construct an example of:

5. a pair of numbers whose sum (difference, product, quotient) is **2.68**; and another, and another.

6. a pair of numbers whose sum (difference, product, quotient) is **¾**; and another, and another.
7. a pair of numbers whose sum (difference, product, quotient) is **–0.2**; and another, and another.

What if not a pair?

Give, find, construct an example of:

8. a set of **three** numbers whose sum is 0; and another, and another.

What if not numbers?

Give, find, construct an example of:

9. a pair of **shapes** whose areas differ by (sum to) 2; and another, and another.
10. a pair of **equations** whose solutions differ by 2; and another, and another.
11. two sets of **data** whose means differ by 2; and another, and another.
12. two lines whose **gradients** differ by 2; and another, and another.
13. two definite **integrals** that differ by 2; and another, and another.
14. a pair of **complex** numbers whose moduli differ by 2; and another, and another.

Here are some more *another* & *another* starting points:

Give, find, construct an example of:

15. a number that rounds to 3.5 (to the nearest tenth); and another, and another.
16. a shape with a pair of parallel sides; and another; and another.
17. a shape with an area of 7; and another, and another.
18. a shape with a perimeter of 7; and another, and another.
19. a set of numbers whose mean is 5; and another; and another.
20. an event whose probability is ⅙; and another; and another.
21. a spinner for which the probability of a 2 is twice the probability of a 4; and another; and another.
22. a unit fraction which is the sum of two unit fractions; and another; and another.
23. a solid with a volume of 12; and another, and another.
24. average speeds and times which could be used to travel 120 miles; and another; and another.
25. a formula that has the value 7 when a=2 and b=3; and another; and another.
26. a pair of points whose mid-point is (2, 3); and another; and another.
27. an equation of a straight line which passes through (2, 3); and another; and another.
28. a point (x, y), such that $3x + 4y = 32$; and another; and another.
29. a right-angled triangle whose hypotenuse is 5; and another; and another.

30. a point which is a distance three from (4, 5); and another; and another.
31. an angle whose sine is 0.5; and another; and another.
32. a region formed by the x-axis, the y-axis, and a line going through (5, 3); and another; …
33. a sequence whose differences form a G.P; and another; and another.
34. a function which passes through the point (2, 3); and another; and another.
35. a function whose derivative is $2x$; and another; and another.
36. a function whose derivative at $x = 0$ is zero; and another; and another.
37. a sequence whose limit is 2; and another; and another.
38. three forces whose resultant is zero (two, three, four or five forces etc.); and another;….
39. a complex number whose modulus and argument are numerically the same; and another; and another.
40. a logarithm which is equal to $\log_2 24$; and another; and another.
41. a pair of functions for which $f \circ g = g \circ f$ for all x.; and another; and another.
42. a matrix whose determinant is 6; and another; and another.
43. a composite function made up of cos, sin and ln which has a value of 1 or 0 when $x = 1$; and another; and another.

Note that subtle differences in tasks can promote different responses in learners. For example, consider the difference in your response to these prompts:

- **Write down three examples of subtractions.**
- **Write down three subtractions with their answers.**
- **Write down three very different examples of subtractions.**
- **Write down as many different kinds of subtraction as you can think of.**
- **Write down a hard subtraction and an easy subtraction.**
- **Write down a subtraction which you would use to explain the idea to a younger person.**
- **Write down several examples of subtractions.**
- **Write down a subtraction which involves numbers with different numbers of digits.**
- **Write down a subtraction in which one or both numbers have zeros somewhere.**

You may like to think how small changes in the prompts would lead to different responses in each of our activities.

2 Pointing Toward Generality (Particular, Peculiar, General)

Learners can also be led to generalisation by asking for a particular then a peculiar example. Once they develop fluency in locating peculiar examples, they can also be encouraged to try to say in what way an example is generic, and to express the form of a general example. For instance:

Give me an example of a fraction that is equivalent to $\frac{2}{3}$.

Give me a really peculiar example.

Give me a general example.

Thinking about the peculiar can help with thinking about the general. To create more and more peculiar examples we need to double any number for the numerator and treble it for the denominator. For example 2 billion over 3 billion or 2×3.22 over 3×3.22 or 2π over 3π. In the classroom learners can compete to make the examples more and more peculiar – numerator and denominator can be integers, decimals, fractions, algebraic expressions, ...

From here it is a short step to an expression of the general:
'2 times something over 3 times something is a fraction equivalent to $\frac{2}{3}$'
or '$\frac{2x}{3x}$ is equivalent to $\frac{2}{3}$ for any value of x other than zero'.

Here are some more starters for the PPG treatment:

Give me an example, a peculiar example, a general example, of

1. an even number.
2. a number with exactly three factors.
3. a number which leaves remainder 1 when divided by 3.
4. a parallelogram.
5. a fraction equivalent to 0.2.
6. a fraction bigger than 3.
7. a shape with rotational symmetry of order 2.
8. an algebraic fraction which is equivalent to $\frac{a}{b}$.
9. a quadratic with a root of 2.
10. a straight-line graph.
11. an angle whose cosine is 0.5.
12. an odd function.

… any of the 'Give an example …another and another' tasks could be set in the PPG mode.

3 Hard and Easy

Give an example which is really hard or complicated and one which is easy or simple. What makes them hard or easy? The examples learners choose reveal a good deal about what they find difficult, and it is not always the concepts which are the source of difficulty. For example, they may be happy with trig ratios but flounder with the algebraic manipulation involved in using them; they may be happy with the process of solving linear equations involving integers, but fearful of using decimals or fractions.

Give an easy or simple example, and a hard or complicated example of

1. a calculation whose answer is 7.
2. a three digit subtraction.
3. a division with zeros in the dividend.
4. fractions to subtract, or divide.
5. a set of numbers to be placed in order.
6. a word problem.
7. a calculation to be done on a calculator.
8. a directed number calculation.
9. a percentage calculation.
10. a region made up of rectangles whose area and perimeter are to be found.
11. a polygon whose interior angles need to be found.
12. a sequence whose nth term is to be found.
13. a question involving ratio.
14. a formula to be evaluated.
15. an expression to be simplified by collecting like terms.
16. a linear equation to solve.
17. a region made up of arcs and sectors of circles whose area and perimeter are to be found.
18. a probability question involving combined events.
19. a quadratic equation to solve.
20. a pair of simultaneous linear equations to solve.
21. a set of data for which a histogram is to be drawn.
22. an application of the sine rule to solve a triangle.
23. a graph of a function whose inverse is to be graphed.
24. an equation to solve by an iterative method.
25. a polynomial to differentiate.
26. a plane whose vector equation is required.
27. a differential equation to solve.

28. a centre of gravity to be found.

29. a solid of revolution.

'Create a hard example' can be used to assess the scope of learners' understanding, but it is also a valuable learning experience, encouraging them to extend the boundaries of what they think is possible. Furthermore, it is useful for preparing them to recognise which technique is appropriate in a new situation, such as an examination question. Learners used to creating hard examples for themselves are not going to be so easily wrong-footed when the examiner comes up with a question that their teacher wouldn't have thought of.

4 Additional Conditions

Imposing constraints can also drive mathematical thinking. We ask first for an example with one constraint, then repeat this with more constraints added one by one. Each successive constraint can cause learners to think more precisely about the properties of the examples they are generating.

Give me an example of

1. a number which is greater than 2.
   a number which is greater than 2 and also greater than 5.
   a number which is greater than 2 and also greater than 5 and also less than 6.
2. a number which has remainder of 1 when divided by 2
   … and remainder of 1 when divided by 3
   … and remainder of 1 when divided by 5.
3. a fraction which is greater than one half
   … and also its numerator and denominator are each greater than 5
   … and also its numerator and denominator are (or are not) multiples of the same number.
4. a number which is 0.6 when rounded to 1 decimal place
   … and is 0.60 when rounded to 2 decimal places
   … and is 0.600 when rounded to 3 decimal places.
5. a set of numbers whose mean is 5
   … and whose mode is 4
   … and whose median is 3
   … and whose range is 6.
6. (for the brave): … and whose standard deviation is 1.
7. a quadrilateral with at least two right angles
   and whose sides are not all the same length
   and which has reflective symmetry about at least one diagonal.
8. a triangle with height (altitude) 2 units
   … and height (altitude) 1 unit
   … and height (altitude) 3 units.
9. a point (x, y) such that $3x + 4y = 32$
   … and $x = 2y$.
10. a parabola which passes through (0, 1)
    … and passes through (1, 2)
    … and also passes through (0.5, 1).
11. a parabola with a turning point at (3, –2)
    … and passing through (0, 0).
12. a circle which touches the x-axis
    … and y-axis
    … and has radius 5.
13. a sequence whose limit is 1
    … and whose terms are not increasing,
    … and whose terms are not decreasing.

In some cases the constraint restricts us to one solution. We can highlight this whilst extending types of numbers to be considered by giving a sequence of similar questions:

14. a pair of numbers whose sum is 5 and whose difference is 1.
15. a pair of numbers whose sum is 5 and whose difference is 4.
16. a pair of numbers whose sum is 5 and whose difference is 0.3.
17. a pair of numbers whose sum is 5 and whose difference is 7.

We might also successively reduce the options:

18. a hexagon with six lines of symmetry;
    with only two lines of symmetry;
    with only one line of symmetry.

or we could successively increase the options:

19. a hexagon with only one line of symmetry;
    with only two lines of symmetry;
    with six lines of symmetry.

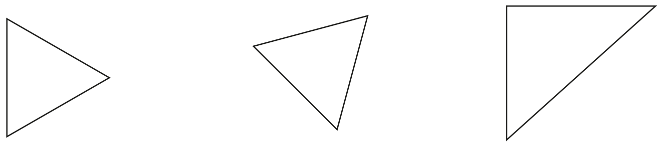
5 Comparing/Contrasting Three

Here the idea is to compose groups of three examples with some similarities and differences. By systematic comparison the exploration of similarities promotes generalisation, and the exploration of differences promotes a deeper and more extended appreciation of the concept.

Choose any two of these three. In what ways are they the same as each other, and different from the third?

1. 7.69   7.74   7.75

2. 2, 4, 6, 8, …   2, 5, 8, 11, …   3, 5, 7, 9,…

3. 

4. 1:3   1:4   2:6

5.

size	frequency
1	4
2	6
3	10
4	6
5	4

size	frequency
4	3
5	7
6	10
7	7
8	3

size	frequency
4	4
5	10
6	8
7	6
8	2

6. $2x - 7 = 15$ $x^2 = 121$ $2x + 6 = 121$

Alternatively, ask learners to generate threes:

7. Give an example of a number which rounds to 7.7
and another which is similar in some way
and another which is different in some way to the first two.

After each of the following the teacher would ask "and another which is similar in some way, and another which is different in some way to the first two."

8. Give an example of a word problem …
9. Draw an example of a polygon …
10. Give an example of a question involving percentages …
11. Give an example of a question involving ratio …
12. Give an example of an event whose probability is one half …
13. Give an example of a sequence …
14. Give an example of a formula to be evaluated …

15. Give an example of an equation of a straight line …
16. Give an example of a triangle to be solved using trigonometry …
17. Give an example of a triangle to be solved using Pythagoras' Theorem …
18. Give an example of a quadratic equation …
19. Give an example of a theorem …
20. Give an example of a vector …
21. Draw the graph of a function …
22. Give an example of a function which can be integrated by substitution …
23. Give an example of an equation of a plane …
24. Give an example of three forces maintaining a body in equilibrium …

Again this exercise provides a richer learning experience when learners share their ideas and the less adventurous seek to emulate their more expansive peers. The teacher's role is to help draw out the finer points of difference and similarity.

6 Confounding Expectations

Once learners know something about a concept, they can develop very fixed ideas about what is possible. A creative way of jolting narrow interpretations is by challenging learners to create something which sounds impossible. It only sounds impossible if it is outside current expectation. The following examples all have hidden assumptions about learners' expectations.

Give, find, construct an example of:

1. a triangle with no edges parallel to the sides of your page.
2. a number that stays the same when multiplied by 10.
3. a hexagon with no lines of symmetry.
4. a symmetrical shape which is not regular.
5. a fraction that is equal to a whole number.
6. a pair of fractions whose product is greater than 1.
7. a pair of numbers whose sum is greater than one of them and less than the other.
8. a pair of numbers whose product is greater than one of them and less than the other.
9. a number less than 5 whose square is greater than 25.
10. an equation whose solution is not a whole number.
11. an equation of a straight line which does not depend on x.
12. an event with two outcomes where the probability of each is not one half.
13. two events, where the probability of both happening is not the product of the separate probabilities.
14. two angles subtended at the circumference of a circle by the same chord, but not equal.
15. a triangle, the centre of whose circumcircle is outside the triangle.
16. a pair of simultaneous linear equations which have no solution.
17. an angle whose sine is 0.5 and which is greater than 90 degrees.
18. a function whose graph does not go through the first quadrant.
19. a function with no inverse.
20. an exponential function which does not increase without bound.
21. a function of the form $f(x) = x^{(1/n)}$ such that your graph plotter will/will not plot it for all values of x in a specified interval from a to b.
22. a 2 by 2 matrix with no inverse.

7 Impossible Constructions

If every task has a solution, learners may form the impression that any question can be answered. It is useful therefore to intersperse tasks with ones which truly are impossible, not just sound impossible as in the previous section. The pedagogical purpose is not only to remind learners that they need to be alert and questioning, but also to stimulate them to explain why it is impossible. In each case, learners can be asked to make up more impossible tasks like it, as a step towards justifying why it is impossible, and even characterising the conditions under which it is either possible, or impossible.

Give, find, construct an example of:

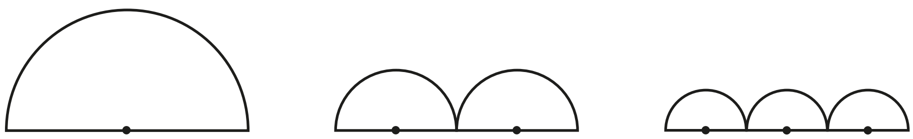
1. a triangle with sides 3, 4 and 8.
2. an odd number and an even number whose sum is not an odd number.
3. a multiple of 6 that is not a multiple of 3.
4. a whole number ending in 2 and a whole number ending in 5 whose product does not end in 0.
5. a number, in decimal form, less than 4 and greater than 3.9 which does not have the digit 9.
6. a number in decimal form that cannot be written as a fraction.
7. a sum of two fractions that is not a fraction.
8. a negative prime number.
9. a shape with rotational symmetry of order 3 and exactly two lines of reflective symmetry.
10. a tessellation of the plane with squares and regular hexagons.
11. a rotation which is equivalent to a reflection for all objects.
12. an event with a probability greater than 1.
13. a square number with an even number of factors.
14. a solid shape in which one vertex has six equilateral triangles meeting.
15. a solid shape all of whose cross-sections are triangular.
16. a ratio question that does not involve multiplication or division.
17. two non-parallel planes which never meet.
18. a polynomial in x whose graph has an infinite gradient for some finite value of x.
19. a solution to the equation $\sin^2 x = 6$.
20. a quadratic with one real and one complex root.
21. a quartic function with exactly three real roots.

8 On The Spot Generalisation

Here learners are invited to generalise from a single example. These are generic in the sense that they stand-in for all other examples: the general can be seen through the single particular. Once 'spotted' learners can be asked to give other examples and explain why the conjectured generalisation is valid.

1. If you are **5**th in the queue then there are **4** people in front of you.
2. $\mathbf{26} + 9 = 3\mathbf{5}$
3. $\mathbf{7}{:}37$ pm $= \mathbf{19}{:}37$
4. 

5. $1456\mathbf{7} - 44\mathbf{7} = 1412\mathbf{0}$
6. $\mathbf{36} \times 10 = \mathbf{36}0$ and $0.\mathbf{36} \times 10 = \mathbf{3}.\mathbf{6}0$
7. 36 times 42 = 42 times 36 and $36 + 42 = 32 + 46$
8. $3 - (3 - 2) = 2$
9. $3 \times (3 - 2) = 3$
10. $(3 - 1) - (3 - 2) = 2 - 1$
11. $10 - {}^{-}\mathbf{7} = 1\mathbf{7}$
12. If there are **5** equally likely events the probability of each is $1/5$
13. If we start at 5 and count in 3s the 7th number is $5 + \mathbf{6} \times 3$
14. $(3 + 5)^2 = 3^2 + 5^2 + 2 \times 3 \times 5$
15. $(x + \mathbf{3})(x + \mathbf{4}) = x^2 + (\mathbf{3} + \mathbf{4})x + \mathbf{3} \times \mathbf{4}$
16. Perimeters:



17. $3^{\circ} = 1$
18. $28/99 = 0.2828282828\ldots.$
19. $\sin 20^{\circ} = \sin 160^{\circ}$
20. $y = (2/3)x$ is perpendicular to $y = -(3/2)x$
21. $3y + 2x = 0$ is perpendicular to $2y - 3x = 0$ and to $2y - 3x = 7$

Although we often discourage learners from generalising from only one example, it is nevertheless what mathematicians do when they can see the underlying structure clearly. Such generalisations can be treated as conjectures to be justified.

9 Open and Closed Questions/Exhaustive Lists

Open questions invite answers which have something in common, that is, that they answer the question. Sometimes there are an infinite number of answers and sometimes we could create the exhaustive list. So what is ‘general’ about all the alternatives? What else do they all have in common apart from answering the original question? In what ways can we express the generality?

List and/or describe in another way …

1. the pairs (the collections) of numbers which end in a 5 when you add them.
2. the pairs of whole numbers which have a product which ends in zero.
3. the numbers which give a remainder of two when you divide by 5.
4. the rectangles which have an area of 12.5 cm^2.
5. the parallelograms which have a base 6 cm and area 24 cm^2.
6. the sequences which have a third term of 10.
7. the ratios which are equivalent to 2:3.
8. the pairs of numbers which have squares which sum to 1.
9. the numbe hich have exactly three factors.
10. the sets of four numbers which have a range of 2 and a median of 5.
11. the sets of data which have symmetrical bar charts.
12. the quadrilaterals which have their diagonals bisecting/intersecting at right-angles.
13. the class of triangles which have two sides whose lengths are in the ratio 3 : 2, in terms of the ratios of the length of the third side to the lengths of the other two sides.
14. the centres of circles which have two given lines as tangents.
15. the straight lines which go through the point (4, 5).
16. the graphs which depict a journey of 1 km along a straight road for 10 minutes, without going backwards.
17. the values of b for which $x^2 + bx + 1$ has real roots.
18. the factors which $b(x - 2)(ax^2 - 2ax + a)$ and $(x^2 - 3x + 2)(ax - a)(bx + 2b)$ have in common.
19. the quadratics which go through two specified points, say (–1, 2) and (3, –1).
20. the angles which have their sine lying between $1/2$ and $1/\sqrt{2}$ inclusively.
21. the arithmetic progressions which have 4 and 6 as two of their terms.

Classroom Activities

10 Always, Sometimes, Never True

This type of prompt focuses attention on whether a statement of a general rule is always true, never true, or sometimes true. Learners can construct instances to illustrate when it is or isn't true, and more generally, construct explicit conditions under which it is, and isn't, true.

1. 7 + 5 – 6 + 2 is always 4.
2. All numbers in the 5 times table end in a five.
3. A number less than ten plus a number less than ten = a number less than ten.
4. A number in the second column of a hundred square + a number in the third column of a hundred square = a number in the fifth column of a hundred square.
5. An even number ÷ an even number = an even number.
6. A decimal – a decimal = a whole number.
7. To multiply a number by ten put a 0 on the end.
8. Division always makes smaller.
9. Rectangles with the biggest areas have the biggest perimeters.
10. For any positive number there is a positive number which is bigger/smaller.
11. A quadrilateral with four equal sides is a rhombus.
12. A hexagon with equal sides is regular.
13. $x + 7 = 10$
14. $x + y = 6$
15. $x + 3.6 = y + 3.6$
16. $3x = 24$ and $x + y = 16$
17. The reciprocal of the reciprocal of a number is the number itself.
18. Squaring a number makes it larger.
19. The square of a square root of a number is itself.
20. The square root of the square of a number is itself.
21. The square on the largest side of a triangle is equal to the sum of the squares on the two smaller sides.
22. Between every two numbers there is a rational number.
23. Between any two real numbers there is a rational number.
24. The mean of a frequency distribution is within one standard deviation of the mean of the extremes.
25. $\text{Sin}2x = 2\sin x$

Any of the examples suggested in 'confounding expectations' or 'impossible constructions' could be adapted to be used here. For instance:

26. the sum of an odd number and even number is an even number.
27. a number multiplied by 10 is not the same as itself.

11 Odd-One-Out

Here the 'fly in the ointment' can point to the generality of the others. One counter-example might be the 'grain of sand' that produces the 'pearl' of generalisation. Learners decide which is an odd-one-out then generate more examples which are the like the other two, and more which are like the odd one. They could also find reasons for different ones to be the odd one out. Once the seed is sown there is further scope to extend this activity by inviting learners to generate their own lists.

1. Which sequence is an odd one out, and why?
   2, 5, 8,11, …
   6, 9, 12, 15, …
   7, 10, 13, 16, …
   34, 37, 40, 43, …
   -4, -1, 2, 5, …

2. Which calculation is an odd one out, and why?
   $3 \times 10 = 30$
   $31 \times 10 = 310$
   $423 \times 10 = 4230$
   $0.3 \times 10 = 3$
   $1111 \times 10 = 11110$

3. Which calculation is an odd one out, and why?
   $\frac{2}{3} \times \frac{5}{7} = \frac{10}{21}$
   $\frac{5}{4} \times \frac{3}{14} = \frac{15}{56}$
   $\frac{4}{3} \times \frac{6}{7} = \frac{8}{7}$
   $\frac{1}{13} \times \frac{7}{9} = \frac{7}{117}$
   $\frac{7}{3} \times \frac{7}{3} = \frac{49}{9}$

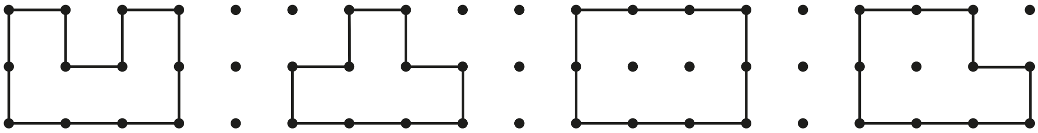
4. Which calculation is an one out, and why?
   $\frac{2}{5} + \frac{3}{5}$
   $\frac{11}{16} + \frac{5}{16}$
   $\frac{1}{9} + \frac{8}{9}$
   $\frac{4}{17} + \frac{12}{17}$
   $\frac{4}{11} + \frac{7}{11}$

5. Which mean of these sets is an odd one out, and why?
   1, 3, 10
   2, 6, 7
   -1, 1, 15
   3.5, 4.2, 7.3
   $\frac{135}{13}, \frac{8}{13}, 4$

6. Which point is an odd one out, and why?

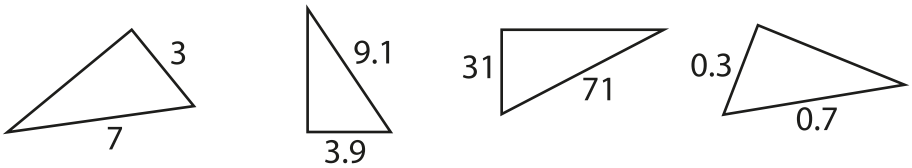
   (3, 7) (6, 13) (-2, -3) (0, 2) (10, 21)

7. Which shape is an odd one out, and why?



8. Which function is an odd one out, and why?
   $y = x^2$
   $y = \cos x$
   $y = x^3$
   $y = x^4$

9. Which is an odd one out in the task to find the unknown angles in these right-angled triangles, and why?



10. Which expression is an odd one out, and why?
    $x^2 + 7x + 12$
    $x^2 + 7x + 13$
    $x^2 + 7x + 14$
    $x^2 + 7x + 15$
    $x^2 + 7x + 16$

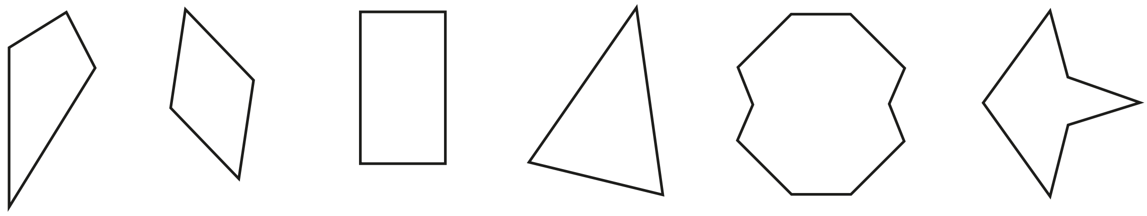
11. Which expression is an odd one out and why?
    $x^2 + 6x + 5$
    $x^2 - 5x + 6$
    $x^2 + 6x - 5$
    $x^2 + 7x + 6$
    $x^2 - 7x + 6$

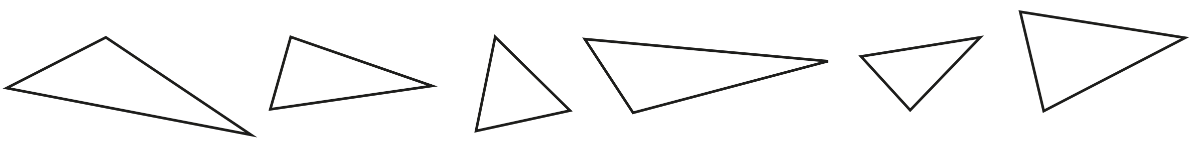
12. Which problem is an odd one out, and why?
    How many DVDs can be purchased for £70 at £13.99 each?
    What is the cost of 5 text messages at 10p per message?
    How many eggs are in 3 dozen?
    How many complete revolutions does the minute hand of a clock make between 3:25 and 14:25?

Learners might find several ‘odd ones out’ in each of these, but the underlying idea is to focus on particular differences and similarities. However, we recommend not treating them as if there are unique answers.

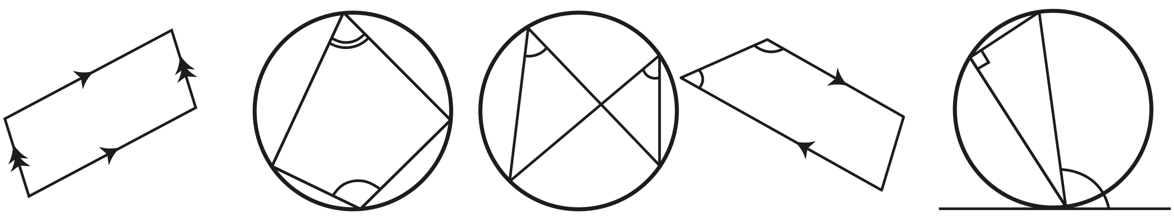
12 Sorting

Two sets of objects which have different properties are mixed together. The task is to partition into the two sets then add some more elements to each. Learners need to recognise the general in the particulars and then create more examples. There may be more than one way to partition so again do not expect unique answers – indeed encourage diverse thinking.

1. 1, 2, 3, 4, 5, 6, 7, 8
2. 353, 378, 451, 502, 437, 549, 450, 449 (suggestion: consider rounding)
3. 
4. $20 \div 3$ $23 \div 3$ $25 \div 3$ $14 \div 3$ $7 \div 3$ $2 \div 3$ $1 \div 3$
5. $1/2$ $3/2$ $3/4$ $9/100$ $100/99$ $999/100$ $51/7$
6. 5% of 12 15% of 4 12% of 5 10% of 8 4% of 20 8% of 10 2% of 30
7. $\frac{4}{5}$ of $\frac{7}{24}$ $\frac{14}{5}$ of $\frac{1}{5}$ $\frac{2}{5}$ of $\frac{3}{5}$ $\frac{2}{5}$ of $\frac{3}{5}$ $\frac{2}{5}$ of $\frac{3}{5}$ $\frac{2}{3}$ of $\frac{7}{30}$ $\frac{7}{10}$ of $\frac{2}{9}$
8. $2x + 3 = 10$ $3x + 2 = 10$ $2x + 53 = 60$ $6x + 3 = 19$
   $3x + 9 = 17$ $2x + 7 = 14$ $4x + 1 = 15$
9. $y = 2x + 5$ $5x - y + 3 = 0$ $3y = 9 + 6x$ $3y - 15x = 9$
   $5y = 25+25x$ $4x - 2y + 10 = 0$ $y = 5x + 3$
10. Sort these triangles according to where their altitudes meet



11. Sort these diagrams



13 Ordering

Placing some similar given objects in order can reveal differences and similarities. These generated sequences can then be extended and explored. The ordering can highlight the ranges of possible change in certain of their attributes. Sometimes the teacher can state the dimension of variation which decides the order; sometimes learners can decide for themselves how ordering might be done, and state this. In these examples we leave it to the reader to provide the raw material.

1. Order numbers according to how close to 1 they are.
2. Order the questions in an exercise so that they progress from easiest to hardest in the learner's opinion.
3. Order triangles according to the size of their largest (smallest) angle.
4. Re-order triangles according to the ratios of their longest side to their shortest sides.
5. Order and re-order a collection of fractions according to the size of the numerator, the size of the denominator, and their value.
6. Order a sequence of events according to their likelihood.
7. Order rectangles according to their perimeter, then re-order according to their area.
8. Order numbers according to how many factors they have.
9. Order some quadrilaterals and learners say how they have ordered them.
10. Order some polygons in two different ways and learners say how these differ.
11. Order and re-order three or more sets of data according to their means, medians, modes.
12. Order graphs according to how steep they are on a given interval.
13. Order a set of linear algebraic expressions according to what their value would be for certain values of x. Explore how the order might change for different values of x.
14. Order a given sequence of statements to construct an argument, or a proof.
15. Order a set of polynomials according to the number of times they would have to be differentiated to reach a constant.
16. Order a set of polynomials according to how many times you would have to take differences before you reached a constant difference.
17. Order sequences according to how rapidly they converge (or grow to infinity).

14 Equivalent Statements

Statements may be logically equivalent, or they may be the same when rearranged according to mathematical rules. It is important that learners recognise many different ways of expressing the same thing. The urge to manipulate symbols arises from trying to see why different expressions are actually equivalent (always give the same result).

1. Give a statement which is equivalent to '3 is less than 5'.
2. Show that 'one more than a multiple of' is the same as having a 'remainder of 1 on dividing by'.
3. Give a statement which is equivalent to 'a is a multiple of b'.
4. Give a statement which is equivalent to 'all squares are rectangles'.
5. Give a statement which is equivalent to 'three specified lengths can be the edges of a triangle'.
6. Show that 'the sequence goes up in threes' is equivalent to 'the terms of the sequence have the same remainder on dividing by 3'.
7. Give a statement which is equivalent to '$x + 5 = 2x - 7$'.
8. Give a statement which is equivalent to '$0.\dot{9} = 1$'.
9. Give a statement which is equivalent to 'area of a triangle is half the area of a rectangle on the same base with the same height'.
10. Give a statement which is equivalent to 'a circle has an infinite number of sides'.
11. Give a statement which is equivalent to 'tan 45° = 1'.

15 With and Across the Grain

The idea here is that learners can get caught up in generating answers to individual technical questions and fail to grasp the general idea with which they are supposed to be working. For example, they can follow all the steps of finding an angle in a right-angled triangle, given two sides, yet fail to develop an understanding of what sine, cosine or tangent mean. By constructing a sequence of repetitive tasks and then getting learners to reflect on them, the teacher draws attention to the main mathematical issue.

By becoming aware of what is changing and what is staying the same, the form of each statement can be discerned, and the pattern of the changing numbers can be extended. Expressing the general pattern in words and/or symbols develops awareness of the role of generalisation in learning mathematics.

1. Extend the following table upwards and downwards (going with the grain). What does each row tell you?

7 x 3 = 21	3 x 7 = 21	21 ÷ 7 = 3	21 ÷ 3 = 7
7 x 4 = 28	4 x 7 = 28	28 ÷ 7 = 4	28 ÷ 4 = 7
7 x 5 = 35	5 x 7 = 35	35 ÷ 7 = 5	35 ÷ 5 = 7

2. Extend the following table upwards, downwards and to both sides (going with the grain). Predict the entry, say, 35 cells to the left and 6 cells up from the marked cell.

			4 – 1 = 3	5 – 1 = 4	6 – 1 = 5		
			4 – 2 = 2	**5 – 2 = 3**	6 – 2 = 4		
			4 – 3 = 1	5 – 3 = 2	6 – 3 = 3		

Note that subtraction could be replaced by other operations or functions involving two things varying.

3. Extend each of the following rows both forwards and backwards (going with the grain), then consider what each statement is saying in general (going across the grain):

$2 \times (10 + 3) = 2 \times 10 + 2 \times 3$	$2 \times (20 + 3) = 2 \times 20 + 2 \times 3$	$2 \times (30 + 3) = 2 \times 30 + 2 \times 3$
$3 \times (3 - 2) + 1 = (3 - 1)^2$	$4 \times (4 - 2) + 1 = (4 - 1)^2$	$5 \times (5 - 2) + 1 = (5 - 1)^2$
$(4 - 1) \times (4 + 1) = 4 \times 4 - 1 \times 1$	$(5 - 1) \times (5 + 1) = 5 \times 5 - 1 \times 1$	$(6 - 1) \times (6 + 1) = 6 \times 6 - 1 \times 1$
$2 \times (2 + 2) + 1 = (2 + 1)^2$	$3 \times (3 + 2) + 1 = (3 + 1)^2$	$4 \times (4 + 2) + 1 = (4 + 1)^2$
$(3 + 2) \times (3 - 1) = 3^2 + 3 - 2$	$(4 + 2) \times (4 - 1) = 4^2 + 4 - 2$	$(5 + 2) \times (5 - 1) = 5^2 + 5 - 2$

16 Burying the Bone

In order to recognise appropriate techniques in exams or in the midst of challenging problems, it helps to be familiar with the range of possibilities, with the complexities which might hide or obscure the relevance of the technique. Consequently it helps learners to try to obscure or complexify for themselves. This also reveals the dimensions-of-possible-variation of which they are aware. Other learners can then try to reconstruct or solve the problem.

Construct an example which makes it difficult to see immediately that:

1. a product of fractions actually simplifies to 2.
2. the sum of some fractions is actually ⅓.
3. a pair of angles are vertically opposite angles of the same size.
4. the mean of the set of numbers is actually 3.5.
5. the product of two particular numbers is a perfect cube.
6. the answer to a specified calculation is a very small number.
7. two expressions have a specified expression as their only common factor.
8. the solution to an equation with three separate occurrences of x is actually $x = 2$.
9. the graphs of a pair of simultaneous equations will intersect at (-2, –3).
10. a difference of two squares is needed to factor the expression.
11. a rational expression simplifies to x/y, and another which simplifies to $(x+y)/(x-y)$

Sometimes it is effective to specify the complexity in some way:

12. Construct an arithmetic calculation whose answer is –3, which uses exactly three multiplications and an odd number of subtractions.
13. Construct a region from rectangles which requires the use of at least three sub-rectangles to work out the area from the given information.
14. Construct a configuration of line segments which requires three uses of Pythagoras to work out all the lengths from the given information.
15. The overall value is 1 (or x/y) for an expression for which the numerator and denominator are themselves fractions.
16. Construct a pair of numbers which require three uses of the Euclidean algorithm to find the greatest common divisor.

The teacher (or learner) might also obscure by giving only partial information:

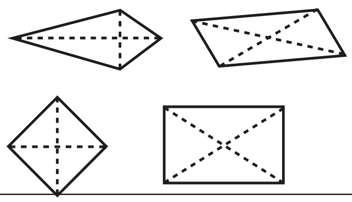
17. Using dynamic geometry software, 'hide' an equilateral triangle by showing only the constructions for its vertices.
18. Using dynamic geometry software, 'hide' a triangle, but reveal three from amongst its medians, altitudes, angle bisectors and right-bisectors, and try to reconstruct the triangle.

Using 'thinkers' in shape and space

In this section we show how the preceding ideas can be used in any topic throughout your teaching - not just in starters and plenaries and not as 'bolt-on' extras, but as the core of your teaching of a topic. We are also going to illustrate how each type of task generates slightly different activity in the learner. To do this we have created a task sequence which could progress, over several years, to geometrical proof, transformational geometry and trigonometry. So that you can cross-reference task-types easily we have used them in the order they appear in this booklet, and we have used every type. In practice, any types can be used in any order, depending on the teacher's intentions.

	Thinker type	*Why you might use this and what might happen*
1	Give me an example of a triangle (perhaps on small whiteboards) ... and another ... and another	Learners show how their notion of triangle is constrained by standard-looking triangles with the base parallel to the bottom of the page. But this prompt does not only give information for teacher assessment at the start of a topic; it also encourages extension of the learners' example spaces by pushing them beyond their first idea... they have to explore further. They thus generate the raw material for the lessons and unusual ones can be discussed. Very young learners, when asked to 'draw another', may vary the size and the colour, not the shape.
2	Can you make one with sides which measure 6, 9, 13 cm? (using dynamic software and circles to fix the lengths of a particular triangle)	Learners experience the problems of trying to make the ends meet up; use of circles to 'fix' lengths solves the problem! Here, learners' frustration is being generated to create a need for a better construction method. Younger learners could be asked to draw particular 'special' triangles instead, such as 'an isosceles triangle which is not equilateral'. It is easy to underestimate what young learners can do with ICT tools.
	Draw a triangle with one very long side and one very short side (or another really 'peculiar' one)	This prompt extends the notion of 'normal' triangle to include 'peculiar' ones; it extends the learners' example spaces beyond the obvious. It may also throw up 'impossible' combinations of lengths, and hence the beginnings of statements of the triangle inequality, i.e. that the sum of two shorter lengths has to be greater than the third length.
	How would you help someone else know whether a shape is a triangle or not?	This kind of prompt follows naturally from box 1 when a teacher notices that no one is offering certain kinds of example. Learners generate their own definitions of 'triangle' which work for all kinds of example, i.e. a generality, by thinking of instructions for others. This could lead to discussion about how little information is needed, and equivalent statements (see box 14).

	Thinker type	*Why you might use this and what might happen*
3	Find triangles which are easy or hard to draw using circles to fix the lengths.	Younger learners could use a set of sticks of different lengths to make 'easy' or 'hard' triangles. Their answers to 'why is this hard?' and 'why is this easy?' will be articulations of mathematical reasoning. Learners' interpretations of 'easy' and 'hard' not only inform the teacher of the scope of their knowledge, but also pushes them beyond the obvious, and implicitly, to evaluate their learning. They have to be able to 'do' their own examples; some learner-generated questions turn out to be insoluble.
	Find quadrilaterals which are easy/ hard to make using two triangles.	Learners could create their own set of triangles to use for this, either physical, so that they get a tactile prompt to remember what they have done, or electronic. This generation of quadrilaterals should produce a wide variety.
4	Find some lengths of triangle sides which produce right-angled triangles.	If they use a set of integer-length sides, the 3, 4, 5 triangle will soon appear. If there has been earlier discussion about similarity (such as might arise from prompt 1 above), then others can be generated by scaling. Using software they could 'fix' a right-angle and create triangles around it with non-integer sides. The idea of fixing one feature and playing with the amount of variation possible with other features is more powerful than trying randomly chosen small integers and seeing what triangles result. This could be demonstrated, or discussed when it arises.
	Find two triangles which, when put together a certain way, produce a concave quadrilateral	Lots can develop from this, such as learner-generated definitions of 'concave', arguments about angles, how the interior angle-sum of polygons can be found from the number of triangles contained, and so on. A writing frame can be given to model the argument, 'For the quadrilateral to be concave, the two triangles must have ....'; 'If two triangles have ..., the quadrilateral made by ..., must be concave because ....'
5	Create a polygon by joining several suitable triangles; ... one which is similar in some way; ... one which is different in some way to the first two	Moving from triangles and quadrilaterals to other polygons, this activity might produce a range of conjectures about angle-sums, symmetry, use of names and properties, and so on. Whole-class sets of the polygons created can be used for further sorting, classifying and defining. Note the difference between being 'mathematically similar' (scaled) and 'alike' through sharing some features, and how being mathematically similar is a special case of being alike in some way. This prompt extends the prompt in box 1 by compelling learners to look at characteristics rather than whole objects. Whatever they choose, their understanding of the possible dimensions of variation will be exercised. The second prompt can be dealt with by changing a variable, but the third has to engage with dimensions of variation.
6	Make a polygon which is symmetrical but not regular.	This pair of challenges clarifies definitions, and also clarifies differences between rotational and reflective symmetry. To be precise, the challenge should have stated what kind of symmetry was required, but ambiguity has to be 'felt' before the significance of such precision is understood.
	Make a polygon which is regular but not symmetrical.	Sometimes what at first looks impossible, turns out to be possible! This prompt links to the next, which focuses on impossible objects.

	Thinker type	*Why you might use this and what might happen*
7	Make a triangle whose medians (or altitudes) do not meet at a point. Make a parallelogram whose opposite angles are not equal.	Learners who have been working with unusual examples of triangles can be encouraged to use a wide range of examples for this. Dynamic software, or 'peoplemaths', is effective here. Learners state their conjectures. Answering 'why?' in this case is a bit hard. If proof is required 'Why can't you?' would be a pre-cursor to a proof argument. Using writing or talking frames helps structure the argument: 'I cannot make ... because ... So a parallelogram must have ... because ...' This prompt follows naturally from the previous one, and also from box 2, which imposes progressive constraints. Some learners will define impossible objects as their 'hard' versions for box 3.
8	What can you say about angles and parallel lines from one set of parallel lines and one transversal?	Learners who are used to dynamic demonstrations can generalise from a static diagram, particularly if they focus on what would change and what would stay the same. Note that a particular shift they have to make is from looking at individual elements to looking at relationships between them. If learners have done a lot of conjecturing, statements of relationships will more easily supplant individual elements as the objects of attention.
9	What angles in the diagram (of parallel lines and a transversal) are the same size? What can be said generally?	Picking out all the equal angles should draw attention to the general places where such angles will be found. Learners articulate their own version of generality. Their descriptions may be diagram specific (such as calling equal sets F or Z angles without realising that sometimes such pairs do not look like Fs or Zs!) in which case 'hard' examples can be sought.
10	Is it always, sometimes or never true that the squares of the two shorter sides of triangles add up to the square of the longest side? Is it always, sometimes or never true that a = b and b = c implies a = c?	Pythagoras' theorem appears as a special case of some inequalities. Lots of examples, in and out of maths, can be used by teachers and learners to generate interest in symbolisations of logical argument. These can then be used to symbolise geometrical arguments.
11	Which is an odd one out? 	There may be several valid answers, and they can be asked to construct their own 'odd one out' sequences, based on geometric properties, to be offered to the whole class.
12	Sort some shapes into whether they have rotational symmetry of order 1, 2 or 4 Sort some theorems into whether or not they depend on parallel line theorems.	This helps sort out 'rotational' from 'reflective' symmetry. Do shapes which have order 4 also have order 2? This is an opportunity to make a class decision about how to treat these distinctions, and what a test-author might think! Historically, many, even most, theorems in school geometry depend (perhaps rather deeply) on the parallel axiom. This could therefore provoke interesting discussion. Treating theorems as objects to be talked about, not just as exercises to prove, helps learners become familiar with the concept of proof.

	Thinker type	*Why you might use this and what might happen*
13	Put some right-angled triangles in order according to the ratios of their sides. Put some geometrical statements into an order which creates a proof	'Order' here is ill-defined, so should generate discussion about orientation and how ratios vary with the size of the angles. Sine, cosine, and tangent can be introduced to make order out of disorder. This creates awareness of how proofs are constructed.
14	Talking about angles, give an equivalent statement to 'equilateral triangles are those which have all sides equal'; or give an equivalent statement to 'right-angles can be made by joining the ends of the diameter of a circle to any point on the circumference'.	With younger learners, equivalent statements should lead them to draw the same shapes or diagrams. With older learners, equivalent statements should provide the possibility of reasoning from one to the other in either direction. Looking at extreme cases (what if the point on the circle coincides with an endpoint of the diameter?) is important mathematically.
15	Calculate the missing angle of these triangles: A= 20 , B = 70, C = ? A = 15, B = 75, C = ? A = 10, B = 80, C = ? ... What do all these have in common? Express this symbolically.	If learners have been busily generating answers to some well-ordered questions they may not see the general pattern of what is being asked; they focus on answers instead of structure. If they are then challenged to look back over their work, across the grain, they might more easily focus on structure.
16	Draw a geometric diagram in which alternate angles are very hard to find but essential to understand the other angles.	Questions in textbooks often involve unpicking complex situations to find an answer. If learners experience for themselves making alternate angles look unusual, perhaps unusually orientated, and existing in a confusion of other angle relationships, they will 'spot' them more easily in future.

We are not suggesting that this sequence of tasks is all that is required to learn, but it goes a long way towards developing mathematical (geometrical) thinking and understanding. The development of these makes teaching and learning mathematics more pleasurable and easier for learners, teachers and helpers. The reader can probably see places in the sequence where there are opportunities to pay attention to vocabulary, fluency, accuracy, oral, practical and physical work, and so on. We want to draw attention to the fact that these opportunities all arise from the power of the questions to promote discussion, activity and argument, and that this power comes from mathematics ... if you start by thinking about mathematical structures, you do not have to impose these pedagogic features artificially on the work, they arise naturally from it.

THEN WHAT?

The following section is a modified version of an article which originally appeared in Mathematics Teaching (Watson 2003). Suppose a question has been asked, responses given and discussed: then what? Every one of the prompts we offer could lead to classroom moments in which the teacher has to decide quickly what to do next. After all, if you give learners the task of generating the ideas and raw material on which the lesson is based you cannot control what happens so much as if you created all the examples yourself. And it makes no sense to get them talking, creating, discussing and then not to use what they produce. Here are some 'then what' moments to consider

A learner is able to describe a general formula which applies to a particular structure and uses it to fill out a table of values for coursework – then what?

A class have shared five different methods of doing a division calculation in a plenary session – then what?

A class have been asked to make up questions for which the answer is 5, they have all done so – then what?

A teacher has asked an open question and written five learners' responses on the board – then what?

In each of these the potential to confuse the form of a strategy with its intended function in terms of learning and doing mathematics is revealed. The question, prompt, or strategy on its own does nothing apart from encourage social participation and open up possibilities for further work.

A learner is able to describe a general formula which applies to a particular structure and uses it to fill out a table of values for coursework – then what?

The learner believes that the table of values is an essential part of the work, but for him the table is of no value as he already sees a generalisation. What is intended to be a tool for generalisation has become some kind of ritual in its own right. Some recent research by psychologists into induction methods shows that systematic generation of values does not necessarily contribute to correct induction – insight and tenacity are more influential. A teacher might celebrate the learner's insight by suggesting finding some unusual values, to relate the formula to the original data, to explore what might happen with negative numbers, to invent structures which generate related formulae. In other words, the 'then what?' is to study the relationships between such formulae and their spatial and graphical representations, their usual and unusual behaviours. What matters is not the formula but how it was constructed from the situation.

A class have shared five different methods of doing a division calculation in a plenary session – then what?

You might have said 'then the bell goes'! Sharing can extend learners' knowledge of what is possible, but they would probably have to do some further work to grasp what is being offered. The focus of such a lesson has shifted from answers to methods but this new focus can dissipate unless the methods are then used to make comparisons. Which methods are appropriate for which kinds of numbers? Which are most efficient? Which are easiest/hardest and why? Which are easiest to record? Does recording help with accuracy? Unfortunately, sharing at the end of the lesson is likely only to fulfil the social function of making people feel successful and involved, or the assessment function of letting the teacher know what has gone on. The form of sharing has been used, but its function in mathematical learning may be absent.

A class have been asked to make up several questions for which the answer is 5, they have all done so – then what?

The beliefs behind this strategy are that making up your own questions helps you answer other people's, and that working backwards helps you understand concepts better than performing algorithms would do. But it is still possible for a learner to make this into a trivial task by choosing obvious and easy options. It is also not clear from research that generating questions and giving them to peers to answer does help answer other people's questions, although it often gives teachers some assessment information and motivates learners to work. Once again, it is possible to confuse the form of question-posing with its function, which is engagement with mathematical concepts. In one class a learner wrote

$5 + 0 =$

$0 + 5 =$

$1 \times 5 =$

$5 \times 1 =$

No one could leave that collection just dangling in the air! For example, one could discuss using '5' as a placeholder for a generality. Another way forward is to make the questions a focus for comparison and discussion, so that learners have to choose their favourite/hardest/most unexpected question from those produced and have to say why.

Another productive approach is to put some constraints on question posing so that learners have to explore concepts in order to produce what is required. For example, 'the answer is 5, there is at least one negative number involved, you cannot use add, you must use every digit at least once, there must be two fractions involved…' and so on. This, of course, turns a quick activity into something which might need a lot of thought and calculator work.

A teacher has asked an open question and written five responses on the board – then what?

The distinction between closed and open questions is not of much help here. Of much more interest is whether a sequence of questions opens-up or closes-down possibilities for a learner, and whether such opening or closing is helping them learn some mathematics. For example, the question 'give me a question whose answer is 5' is very open and consequently can be rather uninteresting. The gradual closing-down suggested above makes it more interesting, more challenging, more mathematical. The closing-down of some possibilities opens-up others. The more closed it gets, the more new possibilities are offered: although there may come a point after which more closure makes things too hard and meets resistance. The purpose of open questions is to encourage thinking and participation, but not all open-ended questions achieve this, while some closed questions can generate a great deal of thought. Participation in unstructured open answers can be merely social. It is the space to produce various answers within constraints which challenges learners to be mathematical. In this case, learners could be asked to sort the answers and say which are most interesting mathematically, and why.

THEN WHAT? – PROMPTING THINKING

In most of our suggestions learners are construing more about mathematics just by exploring the possibilities. In very few of them is the first idea which comes to mind an acceptable answer to the challenge. This is because the tasks generally expect some work to be done on the initial generation of ideas. There is a change of gear from paying attention to the initial 'doing' to thinking about the results of that 'doing' as a class of objects. For example, there could be a shift from getting answers to comparing, contrasting, sorting and generalising methods; a shift from getting one formula to exploring the class of similar formulae; a shift from doing multiplications to looking at multiplicative structures of numbers; a shift from social participation to mathematical participation. The focus is on relating objects or examples through comparing and defining, and thus learning about concepts and classes - the generalities of mathematics. Thus learners are making sense of their mathematical experiences, including 'doing' maths and seeing it done, not in a casual, happenstance sort of way as a by-product of individual and whole class work, but in a structured way which mimics the structures of mathematics.

Of course, this kind of approach creates classroom challenges. How might a teacher manage classes who were noisily discussing and generating ideas? How could a teacher orchestrate all the ideas which emerged? How could a teacher ensure all were participating?

Many of the tasks we propose can be set as written or oral questions, individual or group tasks, so a teacher can make management decisions as appropriate to the circumstances. Most of them require reflection on individual examples or sets of examples in order to be done at all, rather than leaving this kind of thought to plenaries or whole class sessions. Most of them require some kind of articulation of a generality, even if this is only through creating a 'typical' example of some kind. Many of them reveal to the teacher the scope of a learner's knowledge and understanding through the ideas they produce.

The confidence which grows from successful private work can be used to encourage organised classroom discussion, and the confidence which grows from hearing others discuss can be used to support effortful private work. What we have found in our experience is that the kind of questions which require sustained thought will, after an amount of initial encouragement to learners to take safe, supported, risks, lead to more interesting lessons for everyone, including the teacher.

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Number

Shape and Space

Proof

Algebra

Handling data

AS/A2

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